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Some formulas for the restricted r-Lah numbers

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Abstract

The r-Lah numbers, which we denote here by $\ell^{(r)}(n,k)$, enumerate partitions of an $(n+r)$-element set into $k+r$ contents-ordered blocks in which the smallest r elements belong to distinct blocks. In this paper, we consider a restricted version $\ell_m^{(r)}(n,k)$ of the r-Lah numbers in which block cardinalities are at most m. We establish several combinatorial identities for $\ell_m^{(r)}(n,k)$ and obtain as limiting cases for large m analogous formulas for $\ell^{(r)}(n,k)$. Some of these formulas correspond to previously established results for $\ell^{(r)}(n,k)$, while others are apparently new also in the r-Lah case. Some generating function formulas are derived as a consequence and we conclude by considering a polynomial generalization of $\ell_m^{(r)}(n,k)$ which arises as a joint distribution for two statistics defined on restricted r-Lah distributions.

Keywords: restricted Lah numbers, polynomial generalization, r-Lah numbers, combinatorial identities

MSC: 11B73, 05A19, 05A18

1. Introduction

Sequences enumerating certain kinds of finite set partitions in which the smallest r elements are required to belong to distinct blocks are often referred to as r-sequences. Examples that have been studied previously include the r-Stirling numbers [4, 14] of the first and second kind, r-Lah numbers [16] and r-derangement numbers [8, 20]. See also [15] for an r-generalization of the partial Bell polynomials.

A *restricted* version of a counting sequence is one in which the block sizes of the underlying structure are at most a fixed number. Restricted Stirling numbers of both kinds (see, e.g., [6, 10, 11]) have been previously considered in conjunction with incomplete versions of the poly-Cauchy [9] and poly-Bernoulli [10] numbers. See [13] for a common polynomial analogue of the restricted Stirling and Lah numbers.

In this paper, we consider a generalization of the r-Lah numbers, denoted by $\ell_m^{(r)}(n,k)$, by requiring that no block size exceeds m. Note that $\ell_m^{(r)}(n,k)$ reduces to the classical Lah numbers (see, e.g., [12]) when $r = 0$ and $m > n - k$. The limiting case as $m \to \infty$ coincides with the r-Lah numbers studied in [1, 3, 16] and the configurations enumerated in this case are referred to as r-Lah distributions. See [17] for a related polynomial generalization and also [5] where r-Whitney-Lah numbers are introduced. Here, we will study r-Lah distributions with the added restriction that no block size exceeds m. This restriction causes the underlying counting sequence to behave somewhat differently than in the limiting case, which will be manifested by the formulas in the following sections. See [2] where r-Lah distributions are studied in which the block sizes are bounded from below.

This paper is organized as follows. In the next section, we make some preliminary definitions and find a basic recurrence satisfied by $\ell_m^{(r)}(n,k)$ where only the n and k parameters are changing. In the third section, we find some further recurrence formulas for $\ell_m^{(r)}(n,k)$ in which one or both of m and r are changing as well. Taking m large in these formulas recovers prior r-Lah identities in some cases and apparently new identities for these numbers in others. We make use mostly of combinatorial arguments to establish our results, sometimes drawing upon the inclusion-exclusion principle and other times defining a direct bijection between the related structures. An explicit formula is derived in the fourth section in terms of binomial coefficients from which one obtains as a corollary an expression for the generating function. In the final section, a polynomial generalization of $\ell_m^{(r)}(n,k)$ is introduced and some of its properties discussed.

2. Definition and basic recurrence

If m and n are positive integers, then let $[m,n] = \{m, m+1, \ldots, n\}$ for $m \le n$, with $[m,n] = \varnothing$ if $m > n$, the $m = 1$ case of which will simply be denoted by $[n]$. Given $n, k, r \ge 0$ and $m \ge 1$, let $\mathcal{L}_m^{(r)}(n,k)$ denote the set of partitions of $[n+r]$ into $k+r$ contents-ordered blocks (i.e., lists) in which the elements of $[r]$ belong to distinct blocks and all blocks have size at most m. For example, if $n = m = 2$ and $k = r = 1$, then

$$\mathcal{L}_2^{(1)}(2,1) = \{1/23, 1/32, 12/3, 21/3, 13/2, 31/2\}.$$

We will refer to the elements of $[r]$ within a member of $\mathcal{L}_m^{(r)}(n,k)$ as *special* and use this also to describe the blocks to which they belong. Elements of $[r+1, r+n]$ and also the blocks composed solely of these elements will be referred to as *non-special*.

For example, in $15/62/83/47/9 \in \mathcal{L}_2^{(2)}(7,3)$, the blocks 15 and 62 are special, while the blocks 83, 47 and 9 are non-special.

Let $|\mathcal{L}_m^{(r)}(n,k)| = \ell_m^{(r)}(n,k)$. We will denote the limiting case of $\ell_m^{(r)}(n,k)$ as $m \to \infty$ (in particular, if $m > n-k$) by $\ell^{(r)}(n,k)$. Then $\ell^{(r)}(n,k)$ counts the set of all r-Lah distributions of size $n+r$ having $k+r$ blocks (which will be denoted by $\mathcal{L}^{(r)}(n,k)$), as there is no restriction on the block cardinalities when $m > n-k$. When $r=0$ and $m > n-k$, members of $\mathcal{L}_m^{(r)}(n,k)$ coincide with the usual Lah distributions enumerated by [19, Sequence A008297] as there is no restriction also on block membership.

The recurrences below will be based on the following initial conditions. First, let $\ell_m^{(r)}(n,k) = 0$ if any of the parameters are negative or if $0 \le n < k$. If $m=0$, then we will assume $\ell_0^{(r)}(n,k) = 0$ for all n and k if $r > 0$, or if $r=0$ and n and k are not both zero, with $\ell_0^{(0)}(0,0) = 1$. If $r=0$, then an inclusion-exclusion argument gives

$$\ell_m^{(0)}(n,k) = \frac{n!}{k!} \sum_{i=0}^{\lfloor \frac{n-1}{m} \rfloor} (-1)^i \binom{k}{i} \binom{n-mi-1}{k-1}, \qquad n,k,m \ge 1,$$

with $\ell_m^{(0)}(n,0) = \delta_{n,0}$ for all $m \ge 1$. If $n=0$, then $\ell_m^{(r)}(0,k) = \delta_{k,0}$ for all $m,r \ge 1$. Furthermore, the factorial of a negative number will always be taken to be 1 for convenience and the binomial coefficient $\binom{n}{k}$ will be assumed to be zero if n or k is negative or if $k > n \ge 0$.

We now give perhaps the simplest recurrence satisfied by the $\ell_m^{(r)}(n,k)$.

Proposition 2.1. *If $n,m \ge 1$ and $k,r \ge 0$, then*

$$\begin{aligned} k\ell_m^{(r)}(n,k) &= nk\ell_m^{(r)}(n-1,k) + n\ell_m^{(r)}(n-1,k-1) \\ &\quad - \frac{n!}{(n-m-1)!}\ell_m^{(r)}(n-m-1,k-1). \end{aligned} \tag{2.1}$$

Proof. Note that we may assume $k \ge 1$ in (2.1), for it clearly holds if $k=0$. The left side of (2.1) then counts all "marked" members of $\mathcal{L}_m^{(r)}(n,k)$ wherein one of the non-special blocks is marked. Alternatively, in the case when the marked non-special block is not a singleton, one can set aside an element of $[r+1, r+n]$ and then add it at the beginning of the block that is marked within a member of $\mathcal{L}_m^{(r)}(n-1,k)$, yielding $nk\ell_m^{(r)}(n-1,k)$ possibilities. However, one would need to subtract $(m+1)!\binom{n}{m+1}\ell_m^{(r)}(n-m-1,k-1)$ which accounts for the case when adding the extra element results in a block of size $m+1$. On the other hand, if the marked non-special block is a singleton, then there are $n\ell_m^{(r)}(n-1,k-1)$ possibilities, and combining this case with the prior gives (2.1). $\square$

Remark 2.2. The case of (2.1) when $r=0$ and $m \to \infty$ is given in [18, Formula 3.5], where a q-generalization in terms of a statistic on Laguerre configurations is provided.

We observe now the following further special values of $\ell_m^{(r)}(n,k)$. If $m=1$, then $\ell_1^{(r)}(n,k)=\delta_{n,k}$ for all $n,k,r\geq 0$, so it will often be assumed in proofs that $m\geq 2$. If $k=n$, then we have $\ell_m^{(r)}(n,n)=1$ for all $m\geq 1, r\geq 0$. If $k=n-1$ where $n\geq 1$, then considering whether a special or a non-special block has cardinality two implies $\ell_m^{(r)}(n,n-1)=n(2r+n-1)$ if $m\geq 2$. Finally, if $k=n-2$ where $n\geq 2$, then considering several cases concerning the blocks that are not singletons gives the formula

$$\ell_m^{(r)}(n,n-2)=\begin{cases}2r(2r+1)\binom{n}{2}+6(2r+1)\binom{n}{3}+12\binom{n}{4}, & \text{if } m\geq 3;\\ 4r(r-1)\binom{n}{2}+12r\binom{n}{3}+12\binom{n}{4}, & \text{if } m=2.\end{cases}$$

3. Properties of restricted r-Lah numbers

The $\ell_m^{(r)}(n,k)$ are also defined by the following recurrences, where m and/or r is changing as well.

Proposition 3.1. *If $n,m\geq 1$ and $k,r\geq 0$, then*

$$\begin{aligned}\ell_m^{(r)}(n,k)&=\ell_m^{(r)}(n-1,k-1)+(n-1+k+2r)\ell_m^{(r)}(n-1,k)\\&\quad-(m+1)!\binom{n-1}{m}\ell_m^{(r)}(n-m-1,k-1)\\&\quad-r(m+1)!\binom{n-1}{m-1}\ell_m^{(r-1)}(n-m,k)\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\ell_m^{(r)}(n,k)\\&=n!\sum_{i=0}^{k}\sum_{j=0}^{r}\frac{m^j}{i!(n-mi-(m-1)j)!}\binom{r}{j}\ell_{m-1}^{(r-j)}(n-mi-(m-1)j,k-i).\end{aligned}\tag{3.2}$$

Proof. To show (3.1), first observe that there are $\ell_m^{(r)}(n-1,k-1)$ members of $\mathcal{L}_m^{(r)}(n,k)$ such that $n+r$ comprises its own block. Otherwise, the element $n+r$ directly follows some member of $[n+r-1]$ or occurs at the very beginning of a block containing at least one other element. This yields altogether $(n-1+k+2r)\ell_m^{(r)}(n-1,k)$ possibilities if the block sizes were unrestricted. However, one needs to subtract $(m+1)!\binom{n-1}{m}\ell_m^{(r)}(n-m-1,k-1)$ and also $r(m+1)!\binom{n-1}{m-1}\ell_m^{(r-1)}(n-m,k)$ to account for the cases when $n+r$ is added, respectively, to a non-special or special block already containing m elements. Combining the previous cases then gives (3.1).

To show (3.2), let i and j denote, respectively, the number of non-special and special blocks of size m within $\lambda\in\mathcal{L}_m^{(r)}(n,k)$. Then there are

$$\frac{1}{i!}\binom{n}{m,\ldots,m,m-1,\ldots,m-1,n-mi-(m-1)j}(m!)^{i+j}$$

$$= \frac{n!m^j}{i!(n-mi-(m-1)j)!}$$

ways in which to choose and order the elements occupying these blocks of λ, where it is assumed that $mi+(m-1)j \le n$, and $\binom{r}{j}$ choices for the special blocks that are to contain m elements. The rest of the blocks of λ all have size at most $m-1$ and hence there are $\ell_{m-1}^{(r-j)}(n-mi-(m-1)j, k-i)$ ways to arrange the remaining elements of $[n+r]$. Considering all possible i and j gives (3.2). □

Remark 3.2. Letting $m > n$ in (3.1) gives [16, Theorem 3.1], while letting $r = 0$ and $m > n$ in (3.2) gives a Lah analogue of the Stirling number identity found in [11, Proposition 3].

From (3.1), we get the following identity for $n > k \ge 0$:

$$\begin{aligned}\ell_m^{(r)}(n,k) &= \sum_{i=0}^{k}(n+k+2r-2i-1)\ell_m^{(r)}(n-i-1,k-i)\\ &\quad - (m+1)!\sum_{i=0}^{k}\left(\binom{n-i-1}{m}\ell_m^{(r)}(n-m-i-1,k-i-1)\right.\\ &\quad \left. + r\binom{n-i-1}{m-1}\ell_m^{(r-1)}(n-m-i,k-i)\right). \qquad (3.3)\end{aligned}$$

To show (3.3), one can induct on k (starting with $k=0$) and use (3.1) to show that the $(n-1,k)$ case of the identity implies the $(n,k+1)$ case for all n and k.

Theorem 3.3. *If $n \ge 1$, $m \ge 2$ and $k, r \ge 0$, then*

$$\begin{aligned}\ell_m^{(r)}(n,k) = \frac{n!}{k!}\sum_{i=0}^{r}\sum_{j=0}^{i}\sum_{s=0}^{k}\frac{(k-s+j)!}{(n-ms-r+i)!}\binom{k}{s}\binom{r}{j,i-j,r-i}&\\ \times \ell_{m-1}^{(r-i)}(n-ms-r+i,k-s+j).& \qquad (3.4)\end{aligned}$$

Proof. To enumerate the members of $\mathcal{L}_m^{(r)}(n,k)$, first let i denote the number of special elements that start their respective blocks and let s be the number of non-special blocks of size m. Then there are $\binom{n}{m,\ldots,m,n-ms}\frac{(m!)^s}{s!}$ ways in which to select and order the elements within these non-special blocks. Next, let j denote the number of special elements that start non-singleton blocks, where $0 \le j \le i$. Now choose i members of $[r]$ to start blocks and from these select j that are not to form singleton blocks, which can be done in $\binom{r}{i}\binom{i}{j}$ ways (note that the other $i-j$ members of $[r]$ are to occur as singletons). Concerning the remaining $r-i$ members of $[r]$ which do not start blocks, we pick "predecessor" elements from the remaining $n-ms$ members of $[r+1, r+n]$, which can be done in $(n-ms)^{\underline{r-i}}$ ways, where it is implicit that $s \le \min\{k, \lfloor n/m \rfloor\}$.

At this point, we treat each of these $r-i$ special elements, together with their predecessors, as single (special) elements. We form a partition of these elements, together with the $n-ms-r+i$ remaining non-special elements, in which there are $k-s+j$ non-special blocks and all blocks are of size less than m, which can be effected in $\ell_{m-1}^{(r-i)}(n-ms-r+i,k-s+j)$ ways. Let λ denote one of the resulting partitions. Then we choose j of the non-special blocks of λ and add, one-per-block, the j special elements that were selected earlier to start non-singleton special blocks, which can be done in $\binom{k-s+j}{j}j!$ ways. We leave unchanged the remaining blocks of λ. Note that all of the special blocks in the resulting partition ρ actually can have size up to m due to the addition of these elements and to the occurrence of the "double" special elements described above, and thus $\rho \in \mathcal{L}_m^{(r-i+j)}(n-ms,k-s)$ (after standardization). Observe that the non-special blocks of ρ are each of size at most $m-1$ since they correspond to the $k-s$ unselected non-special blocks of λ. Adding the $i-j$ singleton special blocks from above, and also the s non-special blocks of size m, to ρ yields an enumerated member of $\mathcal{L}_m^{(r)}(n,k)$ for the given i, j and s. Note that all members of $\mathcal{L}_m^{(r)}(n,k)$ arise uniquely in this way as i, j and s vary.

Summing over these parameters then implies

$$\begin{aligned}
\ell_m^{(r)}(n,k) &= \sum_{i=0}^{r}\sum_{j=0}^{i}\sum_{s=0}^{k}\binom{r}{i}\binom{i}{j}\binom{n}{m,\ldots,m,n-ms}\frac{(m!)^s}{s!}(n-ms)^{\underline{r-i}}\binom{k-s+j}{j}j! \\
&\qquad\times \ell_{m-1}^{(r-i)}(n-ms-r+i,k-s+j) \\
&= \sum_{i=0}^{r}\sum_{j=0}^{i}\sum_{s=0}^{k}\frac{n!r!}{(r-i)!j!(i-j)!s!}\cdot\frac{(n-ms)^{\underline{r-i}}}{(n-ms)!}\cdot\frac{(k-s+j)!}{(k-s)!} \\
&\qquad\times \ell_{m-1}^{(r-i)}(n-ms-r+i,k-s+j) \\
&= \frac{n!}{k!}\sum_{i=0}^{r}\sum_{j=0}^{i}\sum_{s=0}^{k}\frac{(k-s+j)!}{(n-ms-r+i)!}\binom{k}{s}\binom{r}{j,i-j,r-i} \\
&\qquad\times \ell_{m-1}^{(r-i)}(n-ms-r+i,k-s+j),
\end{aligned}$$

as desired. $\square$

Remark 3.4. Letting m be large in (3.4) gives the following apparently new identity for the r-Lah number $\ell^{(r)}(n,k)$:

$$\ell^{(r)}(n,k) = \frac{n!}{k!}\sum_{i=0}^{r}\sum_{j=0}^{i}\frac{(k+j)!}{(n-r+i)!}\binom{r}{j,i-j,r-i}\ell^{(r-i)}(n-r+i,k+j). \tag{3.5}$$

Considering whether or not a member of $\mathcal{L}_m^{(r)}(n,k)$ contains any special or non-special singleton blocks leads to the following further recurrences.

Theorem 3.5. *If $n,k \geq 0$, $r \geq 1$ and $m \geq 2$, then*

$$\ell_m^{(r)}(n,k) = \sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}\ell_m^{(r-i)}(n,k)$$

$$+\frac{n!}{k!}\sum_{i=0}^{r}\sum_{s=0}^{k}\frac{(k-s+i)!}{(n-ms-r+i)!}\binom{r}{i}\binom{k}{s}\ell_{m-1}^{(r-i)}(n-ms-r+i,k-s+i) \quad (3.6)$$

and

$$\ell_m^{(r)}(n,k)=\sum_{i=1}^{k}(-1)^{i-1}\binom{n}{i}\ell_m^{(r)}(n-i,k-i)$$
$$+n!\sum_{i=0}^{r}\frac{m^i}{(n-k-(m-1)i)!}\binom{r}{i}\ell_{m-1}^{(r-i)}(n-k-(m-1)i,k). \quad (3.7)$$

Proof. We will assume $n,k\geq 1$, as the formulas will be seen to hold also in the case when n or k is zero. To show (3.6), first note that letting $i=j$ in the proof of (3.4) above gives the cardinality of all members of $\mathcal{L}_m^{(r)}(n,k)$ in which no element of $[r]$ forms its own block and that the double sum expression on the right side of (3.6) corresponds to taking only the $j=i$ term in the j-indexed sum in (3.4). Let $\widetilde{\mathcal{L}}_m^{(r)}(n,k)$ denote the subset of $\mathcal{L}_m^{(r)}(n,k)$ whose members contain at least one special singleton block. By subtraction, the difference $\ell_m^{(r)}(n,k)-|\widetilde{\mathcal{L}}_m^{(r)}(n,k)|$ gives the cardinality of all members of $\mathcal{L}_m^{(r)}(n,k)$ containing no special singleton blocks. On the other hand, by the principle of inclusion-exclusion, this cardinality is also given by

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}\ell_m^{(r-i)}(n,k)=\ell_m^{(r)}(n,k)-\sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}\ell_m^{(r-i)}(n,k).$$

Comparing expressions gives $|\widetilde{\mathcal{L}}_m^{(r)}(n,k)|=\sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}\ell_m^{(r-i)}(n,k)$, which implies (3.6).

To show (3.7), note that by similar reasoning, the first sum on the right-hand side counts all members of $\mathcal{L}_m^{(r)}(n,k)$ containing at least one non-special singleton block. To enumerate those $\lambda\in\mathcal{L}_m^{(r)}(n,k)$ that do not contain any, first suppose that exactly i of the special blocks of λ have size m. Then there are $\binom{r}{i}\binom{n}{m-1,\ldots,m-1,n-(m-1)i}(m!)^i=\frac{n!}{(n-(m-1)i)!}\binom{r}{i}m^i$ ways to select and order the elements that comprise these blocks of λ, where $i\leq\lfloor n/(m-1)\rfloor$. Next, we pick k of the remaining elements of $[r+1,r+n]$, which can be done in $\binom{n-(m-1)i}{k}$ ways, and set them aside. We then arrange the rest of the $n-k-(m-1)i$ elements of $[r+1,r+n]$, together with the $r-i$ unchosen elements of $[r]$, according to a member of $\mathcal{L}_{m-1}^{(r-i)}(n-k-(m-1)i,k)$. Then we add the k non-special elements that were set aside to the beginning of the non-special blocks of this partition, one-per-block, according to an arbitrary permutation of $[k]$, to obtain λ. Thus, we get

$$n!\sum_{i=0}^{r}\frac{m^i}{(n-(m-1)i)!}\binom{r}{i}\binom{n-(m-1)i}{k}k!\ell_{m-1}^{(r-i)}(n-k-(m-1)i,k)$$
$$=n!\sum_{i=0}^{r}\frac{m^i}{(n-k-(m-1)i)!}\binom{r}{i}\ell_{m-1}^{(r-i)}(n-k-(m-1)i,k)$$

members of $\mathcal{L}_m^{(r)}(n,k)$ that do not contain a non-special singleton block, which gives (3.7). $\square$

We were unable to find in the literature the r-Lah identities corresponding to the limiting cases of (3.6) and (3.7) as $m \to \infty$. Before stating the next result, let

$$L_m^{(r)}(n,k) = \sum_{j=k}^{n} \binom{j}{k} \ell_m^{(r)}(n,j) \quad \text{and} \quad x^{\overline{k}} = x(x+1)\cdots(x+k-1),$$

for a variable x. We have the following relationship between $\ell_m^{(r)}(n,k)$ and its upper binomial transform $L_m^{(r)}(n,k)$.

Theorem 3.6. *If $n,k,m,p \geq 1$ and $r \geq 0$, then*

$$\sum_{j=1}^{n} (j+p+2r)^{\overline{k}} \ell_m^{(r)}(n,j) = \sum_{i=0}^{n} \sum_{s=0}^{p} (i+s)! \binom{p}{s} \ell^{(r)}(k,i+s) L_m^{(r)}(n,i). \tag{3.8}$$

Proof. First note that $(j+p+2r)^{\overline{k}}$ counts partitions of $[k]$ into $j+p+2r$ labeled, contents-ordered blocks in which some of the blocks may be empty. Let $\mathcal{A}_{n,k}^{(p)}$ denote the set of ordered pairs (α,β) in which $\alpha \in \mathcal{L}_m^{(r)}(n,j)$ and β is a partition enumerated by $(j+p+2r)^{\overline{k}}$ for some $1 \leq j \leq n$. Then $|\mathcal{A}_{n,k}^{(p)}|$ is given by the left-hand side of (3.8). Let $\mathcal{B}_{n,k}^{(p)}$ denote the set of triples (γ,δ,ϵ) such that $\gamma \in \mathcal{L}_m^{(r)}(n,j)$, where i of the non-special blocks of γ are circled for some $0 \leq i \leq j \leq n$, δ is a subset of $[p]$ of size s for some s, and ϵ is a member of $\mathcal{L}^{(r)}(k,i+s)$ in which the non-special blocks can occur in any order. Then $|\mathcal{B}_{n,k}^{(p)}|$ is given by

$$\sum_{j=1}^{n} \sum_{s=0}^{p} \sum_{i=0}^{j} (i+s)! \binom{j}{i} \binom{p}{s} \ell^{(r)}(k,i+s) \ell_m^{(r)}(n,j),$$

which can be rewritten to give the right-hand side of (3.8).

To complete the proof, we define a bijection between the sets $\mathcal{A}_{n,k}^{(p)}$ and $\mathcal{B}_{n,k}^{(p)}$. To do so, let $(\alpha,\beta) \in \mathcal{A}_{n,k}^{(p)}$ and we construct a member of $\mathcal{B}_{n,k}^{(p)}$. Consider the labels of the non-empty blocks among the first j blocks of β (starting from the left) and then those among the non-empty of the next p blocks of β. This determines (possibly empty) subsets S_1 and S_2 of $[j]$ and $[p]$, respectively. Let $\delta = S_2$ and γ be obtained from α by circling the non-special blocks of α corresponding to the subset S_1, where we assume that the non-special blocks of γ are arranged left-to-right in increasing order of smallest elements. To form ϵ, we first create its non-special blocks using the non-empty blocks among the first $j+p$ blocks of β where each element of β is increased by r (note that all of β's blocks are labeled in increasing order from left to right, including the empty ones). To create the q-th special block of ϵ where $1 \leq q \leq r$, we form the word $\rho_1 q \rho_2$, where ρ_1 and ρ_2 denote respectively the ordered contents of the $(j+p+2q-1)$-st and $(j+p+2q)$-th blocks of β (and ρ_1 and ρ_2

are represented using letters in $[r+1, r+k]$). Note that there is no restriction on the block cardinalities of ϵ and that the non-special blocks of ϵ are themselves ordered (since the blocks of β were labeled), whence there are $(i+s)!\ell^{(r)}(k, i+s)$ possibilities for ϵ where $i = |S_1|$ and $s = |S_2|$. One may verify that the mapping $(\alpha, \beta) \mapsto (\gamma, \delta, \epsilon)$ defined by the above construction is a bijection between $\mathcal{A}_{n,k}^{(p)}$ and $\mathcal{B}_{n,k}^{(p)}$, which completes the proof. $\square$

We have the following further r-dependent recurrence.

Theorem 3.7. *If $n, k \geq 0$, $m \geq 1$ and $1 \leq s \leq r$, then*

$$\ell_m^{(r)}(n,k) = n! \sum_{i=0}^{(m-1)s} \frac{\ell_m^{(r-s)}(n-i,k)}{(n-i)!} \left[\sum_{j=0}^{\lfloor \frac{i}{m} \rfloor} \sum_{p=0}^{j} (-1)^j \binom{s}{p, j-p, s-j} \right.$$
$$\left. \times \binom{i+2s-(m+1)j-1}{2s-p-1} (m+1)^p \right]. \quad (3.9)$$

Proof. We start by considering the number t of elements in the first s special blocks within a member of $\mathcal{L}_m^{(r)}(n,k)$, where $s \leq t \leq ms$. Then there are $\binom{n}{t-s}$ choices for the non-special elements within these blocks and $\ell_m^{(r-s)}(n-t+s, k)$ ways in which to arrange elements within the non-special and the final $r-s$ special blocks. Finally, there are

$$(t-s)! \sum_{\substack{\lambda_1+\cdots+\lambda_s=t \\ 1 \leq \lambda_i \leq m}} \lambda_1 \cdots \lambda_s$$

ways in which to arrange the elements in the first s special blocks. To realize this, note that the multi-indexed sum counts all compositions of t with s parts, where the i-th part for each i has size λ_i, $1 \leq \lambda_i \leq m$, and is colored in one of λ_i ways. The i-th special block for each $i \in [s]$ is then to have cardinality λ_i, within which i is to occupy the b_i-th position from the left, where b_i denotes the color assigned to the part λ_i. The $t-s$ non-special elements within the first s special blocks can occur in any order in a left-to-right scan of their contents, which accounts for the $(t-s)!$ factor. Combining the above observations gives

$$\ell_m^{(r)}(n,k) = \sum_{t=s}^{ms} \left(\sum_{\substack{\lambda_1+\cdots+\lambda_s=t \\ 1 \leq \lambda_i \leq m}} \lambda_1 \cdots \lambda_s \right) \frac{n!}{(n-t+s)!} \ell_m^{(r-s)}(n-t+s, k). \quad (3.10)$$

We now simplify the multi-sum in (3.10). To do so, we make use of the inclusion-exclusion principle and sieve out from the set of all (colored) compositions of t having s parts those whose parts are at most m. Consider the number j of parts of size exceeding m; note that $(m+1)j + (s-j) \leq t$ gives $j \leq \lfloor \frac{t-s}{m} \rfloor$, which we denote by u. Then $t \leq ms$ implies $u \leq s$. Thus, we have

$$\sum_{\substack{\lambda_1+\cdots+\lambda_s=t \\ 1 \leq \lambda_i \leq m}} \lambda_1 \cdots \lambda_s$$

$$= \sum_{j=0}^{u} (-1)^j \binom{s}{j} \sum_{\substack{\lambda_1+\cdots+\lambda_s=t-(m+1)j \\ \lambda_i \geq 0}} (\lambda_1+m+1)\cdots(\lambda_j+m+1)\lambda_{j+1}\cdots\lambda_s$$

$$= \sum_{j=0}^{u} (-1)^j \binom{s}{j} \sum_{p=0}^{j} \binom{j}{p} (m+1)^p \sum_{\substack{\lambda_1+\cdots+\lambda_s=t-(m+1)j \\ \lambda_i \geq 0}} \lambda_{p+1}\cdots\lambda_s$$

$$= \sum_{j=0}^{u} (-1)^j \binom{s}{j} \sum_{p=0}^{j} \binom{j}{p} (m+1)^p \binom{s+t-(m+1)j-1}{2s-p-1},$$

where we have used [2, Formula 26] in the last equality. Substituting this into (3.10), and replacing t with $i+s$, gives (3.9). □

When m is large in (3.9), note that $j=p=0$ is required in the two inner sums, which yields the following recurrence for $\ell^{(r)}(n,k)$.

Corollary 3.8 (Nyul and Rácz [16]). *If $n,k \geq 0$ and $1 \leq s \leq r$, then*

$$\ell^{(r)}(n,k) = \sum_{i=0}^{n-k} (2s)^{\overline{i}} \binom{n}{i} \ell^{(r-s)}(n-i,k). \tag{3.11}$$

We conclude this section with the following recurrences which are obtained by considering the nature of the singleton blocks within a member of $\mathcal{L}_m^{(r)}(n,k)$.

Theorem 3.9. *If $n,k \geq 0$, $m \geq 1$ and $r \geq 0$, then*

$$\ell_m^{(r)}(n,k) = \ell_m^{(0)}(n,k) + \sum_{j=1}^{r} \sum_{i=1}^{m-1} (i+1)! \binom{n}{i} \ell_m^{(r-j)}(n-i,k) \tag{3.12}$$

and

$$\ell_m^{(r)}(n,k) = \sum_{j=0}^{r} \sum_{i=0}^{k} \sum_{t=0}^{j} \frac{(n-k+i)!(i+1)^{\overline{t}}}{(n-k-j+t)!} \binom{n}{k-i} \binom{r}{t,j-t,r-j} \times \ell_{m-1}^{(j-t)}(n-k-j+t,i+t). \tag{3.13}$$

Proof. To show (3.12), consider within a member of $\mathcal{L}_m^{(r)}(n,k)$ the smallest $j \in [r]$ if it exists such that the singleton block $\{j\}$ does not occur (note that there are $\ell_m^{(0)}(n,k)$ possibilities if no such j exists). If $i+1$ denotes the cardinality of the block containing j, where $1 \leq i \leq \min\{m-1, n-k\}$, then there are $\binom{n}{i}(i+1)!$ ways in which to select and order the elements belonging to this block. There are thus $\ell_m^{(r-j)}(n-i,k)$ ways in which to arrange the remaining $r-j$ special and $n-i$ non-special elements. Considering all i and j gives (3.12).

To show (3.13), first note that we may assume $m \geq 2$ since the formula is seen to hold when $m=1$. Let $r-j$ and $k-i$ denote the number of special and non-special singleton blocks, respectively, within a member of $\mathcal{L}_m^{(r)}(n,k)$, where $0 \le j \le r$ and $0 \le i \le k$. We select the elements comprising these blocks in $\binom{r}{j}\binom{n}{k-i}$ ways and set them aside. At this point, let us refer to non-singleton special blocks that are to start with a special (non-special, resp.) element as being of type 1 (of type 2, resp.), with type 3 referring to non-singleton non-special blocks. Let t denote the number of type 1 blocks, where $0 \le t \le j$. The special elements in these blocks may be chosen in $\binom{j}{t}$ ways, the set of which we denote by S. We now construct $\lambda \in \mathcal{L}_m^{(r)}(n,k)$ meeting the above specifications with respect to i, j and t. To do so, we first choose $i+j-t$ of the remaining elements of $[r+1, r+n]$ that are to start either one of the $j-t$ blocks of λ of type 2 or one of its i blocks of type 3, which can be done in $\binom{n-k+i}{i+j-t}$ ways, the set of which we denote by T. We place the elements of T aside and then arrange all elements of $[n+r]$ not chosen thus far according to some partition ρ, where ρ (when standardized) belongs to $\mathcal{L}_{m-1}^{(j-t)}(n-k-j+t, i+t)$.

We now choose t of the non-special blocks of ρ in one of $\binom{i+t}{t}$ ways and then add a member of S to the beginning of each of these blocks according to any permutation of S. This produces the t type 1 blocks of λ. Next, we add the elements of T, one-per-block, to the beginning of the remaining $i+j-t$ blocks of ρ (i.e., those that did not receive an element of S), which can be done in $(i+j-t)!$ ways. This gives all of the blocks of λ of type 2 or 3. Appending as singleton blocks the $r-j$ special and the $k-i$ non-special elements set aside above completes the construction of the enumerated partition λ. One may verify that all λ satisfying the given requirements arise uniquely in this manner. By the preceding, the cardinality of such λ is given by

$$\begin{aligned}
&\binom{r}{j}\binom{n}{k-i}\binom{j}{t}\binom{n-k+i}{i+j-t}\binom{i+t}{t}t!(i+j-t)!\ell_{m-1}^{(j-t)}(n-k-j+t, i+t)\\
&\quad = \frac{(n-k+i)!(i+1)^{\overline{t}}}{(n-k-j+t)!}\binom{n}{k-i}\binom{r}{t, j-t, r-j}\ell_{m-1}^{(j-t)}(n-k-j+t, i+t).
\end{aligned}$$

Summing over i, j and t yields all members of $\mathcal{L}_m^{(r)}(n,k)$. $\square$

Remark 3.10. Letting $m \to \infty$ in (3.12) and (3.13) gives further identities for the r-Lah numbers. Letting $r = 0$ in the second of these identities implies

$$\ell^{(0)}(n,k) = \sum_{i=0}^{k} \frac{(n-k+i)!}{(n-k)!}\binom{n}{k-i}\ell^{(0)}(n-k, i), \qquad n, k \ge 0,$$

which can also be shown directly using the $r = 0$ case of (4.4) below. Note that by using the formula from (4.4) in the limiting case of (3.13), one obtains an interesting family of binomial coefficient identities indexed by r.

4. Explicit formula for $\ell_m^{(r)}(n,k)$

We provide a combinatorial proof of the following expression for $\ell_m^{(r)}(n,k)$ in terms of binomial coefficients.

Theorem 4.1. *If $n,k,m \geq 1$ and $r \geq 0$, then*

$$\ell_m^{(r)}(n,k) = \frac{n!}{k!}\sum_{i=0}^{k+r}\sum_{t=0}^{r}(-1)^i\binom{r}{t}\binom{k+r-t}{i-t}\binom{n+2r-mi-t-1}{k+2r-t-1}m^t. \tag{4.1}$$

Proof. Let $C_{n,k,r}^{(m)}$ denote the set of compositions of $n+r$ having $k+r$ parts each of size at most m in which the first r parts are colored such that a part of size x is colored in one of x ways (the remaining k parts are not uncolored). Then we have $\ell_m^{(r)}(n,k) = \frac{n!}{k!}|C_{n,k,r}^{(m)}|$. To see this, given $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_{r+k}) \in C_{n,k,r}^{(m)}$, we distribute the elements of $[n+r]$ in blocks such that $\lambda_1,\ldots,\lambda_r$ correspond to the cardinalities of the special and $\lambda_{r+1},\ldots,\lambda_{r+k}$ to the cardinalities of the non-special blocks (written in any order), where the element $i \in [r]$ is to occupy the position $a \in [\lambda_i]$ (from the left) within its block if a is the color assigned to the part λ_i of λ. Then there are $n!$ ways to arrange the elements of $[n+r]$ as described once λ is specified, and we divide by $k!$ since the non-special blocks are themselves not to be ordered.

Next, we determine the cardinality of $C_{n,k,r}^{(m)}$ and first show

$$|C_{n,k,r}^{(m)}| = \sum_{i=0}^{k+r}(-1)^i\sum_{j=0}^{i}\binom{r}{j}\binom{k}{i-j} \times \sum_{\lambda_1+\cdots+\lambda_{k+r}=n+r-mi}(\lambda_1+m)\cdots(\lambda_j+m)\lambda_{j+1}\cdots\lambda_r, \tag{4.2}$$

where the λ_i are positive in the innermost sum. To do so, first let $C_{n,k,r}^*$ denote the set of compositions of $n+k$ having $k+r$ parts in which the first r parts are colored just as members of $C_{n,k,r}^{(m)}$ were above, where now part sizes are unrestricted and where a (possibly empty) subset of the parts of size $m+1$ or more is circled. Let $C_{n,k,r}^*(i)$ denote the subset of $C_{n,k,r}^*$ containing exactly i circled parts. Then we have $|C_{n,k,r}^{(m)}| = \sum_{i=0}^{k+r}(-1)^i|C_{n,k,r}^*(i)|$. To see this, let members of $C_{n,k,r}^*(i)$ have sign $(-1)^i$. Define a sign-changing involution on $\cup_{i=0}^{k+r}C_{n,k,r}^*(i)$ by identifying the leftmost part of size greater than m and either circling or uncircling it (where the color is preserved, if the part is among the first r). The survivors of this involution comprise the set $C_{n,k,r}^{(m)}$, so to complete the proof of (4.2), it suffices to show

$$|C_{n,k,r}^*(i)| = \sum_{j=0}^{i}\binom{r}{j}\binom{k}{i-j}\sum_{\lambda_1+\cdots+\lambda_{k+r}=n+r-mi}(\lambda_1+m)\cdots(\lambda_j+m)\lambda_{j+1}\cdots\lambda_r. \tag{4.3}$$

To establish (4.3), consider the number j of parts among the first r that are circled within a member of $C^*_{n,k,r}(i)$. Then there are $\binom{r}{j}\binom{k}{i-j}$ possible ways to select the parts that are to be circled. Note that the number of possible ways in which to color the first r parts depends on how many and not which of these parts are circled. Thus, once j is given, we may assume that it is the first j parts that are circled. Once the circled parts of $\lambda \in C^*_{n,k,r}(i)$ are specified, it follows that there are

$$\sum_{\lambda_1+\cdots+\lambda_{k+r}=n+r-mi} (\lambda_1+m)\cdots(\lambda_j+m)\lambda_{j+1}\cdots\lambda_r$$

ways to determine the sizes of all the parts of λ together with the colors of the first r parts. Considering all possible j then implies (4.3) and thus (4.2), as desired.

Now observe that

$$\begin{aligned}
&\sum_{\substack{\lambda_1+\cdots+\lambda_{k+r}=n+r-mi\\ \lambda_\ell\geq 1}} (\lambda_1+m)\cdots(\lambda_j+m)\lambda_{j+1}\cdots\lambda_r\\
&= \sum_{t=0}^{j}\binom{j}{t}m^t \sum_{\substack{\lambda_1+\cdots+\lambda_{k+r}=n+r-mi\\ \lambda_\ell\geq 1}} \lambda_{t+1}\lambda_{t+2}\cdots\lambda_r\\
&= \sum_{t=0}^{j}\binom{j}{t}m^t \sum_{\substack{\lambda_1+\cdots+\lambda_{k+r}=n-k+r-mi-t\\ \lambda_{t+1},\ldots,\lambda_r\geq 1\\ \lambda_\ell\geq 0 \text{ otherwise}}} \lambda_{t+1}\lambda_{t+2}\cdots\lambda_r\\
&= \sum_{t=0}^{j}\binom{j}{t}\binom{n+2r-mi-t-1}{k+2r-t-1}m^t,
\end{aligned}$$

where we have used [2, Formula 26] in the last equality. Thus, by (4.2) and [7, Identity 5.23], we get

$$\begin{aligned}
|C^{(m)}_{n,k,r}| &= \sum_{i=0}^{k+r}(-1)^i\sum_{j=0}^{i}\binom{r}{j}\binom{k}{i-j}\sum_{t=0}^{j}\binom{j}{t}\binom{n+2r-mi-t-1}{k+2r-t-1}m^t\\
&= \sum_{i=0}^{k+r}(-1)^i\sum_{t=0}^{r}\binom{r}{t}\binom{n+2r-mi-t-1}{k+2r-t-1}m^t\sum_{j=t}^{i}\binom{k}{i-j}\binom{r-t}{j-t}\\
&= \sum_{i=0}^{k+r}(-1)^i\sum_{t=0}^{r}\binom{r}{t}\binom{k+r-t}{i-t}\binom{n+2r-mi-t-1}{k+2r-t-1}m^t,
\end{aligned}$$

which implies (4.1). □

Allowing m to be large in (4.1) recovers the following explicit formula for $\ell^{(r)}(n,k)$.

Corollary 4.2 (Nyul and Rácz [16]). *If $n, k \geq 1$ and $r \geq 0$, then*

$$\ell^{(r)}(n,k) = \frac{n!}{k!}\binom{n+2r-1}{k+2r-1}. \tag{4.4}$$

Let $f_{k,m}^{(r)}(x) = \sum_{n \geq k} \ell_m^{(r)}(n,k)\frac{x^n}{n!}$. Multiplying both sides of (4.1) by $\frac{x^n}{n!}$, summing over $n \geq k$ and interchanging summation, we have

$$\begin{aligned}
f_{k,m}^{(r)}(x) &= \frac{x^k}{k!(1-x)^{k+2r}} \sum_{i=0}^{k+r}\sum_{t=0}^{i}(-1)^i\binom{r}{t}\binom{k+r-t}{i-t}m^t x^{mi}(1-x)^t \\
&= \frac{x^k}{k!(1-x)^{k+2r}} \sum_{t=0}^{r}\binom{r}{t}m^t(x^{m+1}-x^m)^t \sum_{i=0}^{k+r-t}(-1)^i\binom{k+r-t}{i}x^{mi} \\
&= \frac{x^k(1-x^m)^{k+r}}{k!(1-x)^{k+2r}} \sum_{t=0}^{r}\binom{r}{t}\left[\frac{mx^m(x-1)}{1-x^m}\right]^t \\
&= \frac{(x-x^{m+1})^k}{k!(1-x)^{k+2r}}(1-(m+1)x^m+mx^{m+1})^r.
\end{aligned}$$

Let

$$f_m^{(r)}(x,y) = \sum_{n \geq 0}\left(\sum_{k=0}^{n}\ell_m^{(r)}(n,k)y^k\right)\frac{x^n}{n!}.$$

Multiplying the last equality by y^k, and summing over $k \geq 0$, yields the following result.

Corollary 4.3. *If $m \geq 1$ and $r \geq 0$, then*

$$f_m^{(r)}(x,y) = \left(\frac{1-(m+1)x^m+mx^{m+1}}{(1-x)^2}\right)^r \exp\left(\frac{x(1-x^m)}{1-x}y\right). \tag{4.5}$$

Consider $L_{n,m}^{(r)}$ defined by

$$L_{n,m}^{(r)} = \sum_{k=0}^{n}\ell_m^{(r)}(n,k)\frac{(-1)^{n-k}}{k+1}, \qquad m \geq 1,\ r \geq 0. \tag{4.6}$$

Note that when $r = 0$, the $L_{n,m}^{(r)}$ provide a Lah analogue to the restricted Cauchy numbers studied in [11], which reduce to the classical Cauchy numbers as $m \to \infty$. Let $L_m^{(r)}(x) = \sum_{n \geq 0} L_{n,m}^{(r)}\frac{x^n}{n!}$. Our next result in the $r = 0$ case is analogous to the one from [11] for restricted Cauchy numbers.

Proposition 4.4. *If $m \geq 1$ and $r \geq 0$, then*

$$L_m^{(r)}(-x) = \frac{(1-(m+1)x^m+mx^{m+1})^r\left(1-\exp\left(\frac{x(1-x^m)}{x-1}\right)\right)}{x(1-x^m)(1-x)^{2r-1}}. \tag{4.7}$$

Proof. Multiplying both sides of (4.6) by $\frac{(-x)^n}{n!}$, and summing over $n \geq 0$, implies

$$L_m^{(r)}(-x) = \sum_{k\geq 0} \frac{(-1)^k f_{k,m}^{(r)}(x)}{k+1} = \int_0^1 \sum_{k\geq 0} f_{k,m}^{(r)}(x)(-y)^k dy = \int_0^1 f_m^{(r)}(x,-y)dy$$
$$= \frac{1-x}{x(1-x^m)}\left(1-\exp\left(\frac{x(1-x^m)}{x-1}\right)\right)\left(\frac{1-(m+1)x^m+mx^{m+1}}{(1-x)^2}\right)^r,$$

by (4.5), which gives (4.7). □

5. Polynomial generalization

In this section, we briefly discuss a polynomial generalization of the sequence $\ell_m^{(r)}(n,k)$ based on a pair of statistics on $\mathcal{L}_m^{(r)}(n,k)$. Given a block $\mathcal{B}$ of $\lambda \in \mathcal{L}_m^{(r)}(n,k)$, an element $i \in \mathcal{B}$ is said to be a *left-to-right minimum* if there exists no j to the left of i in $\mathcal{B}$ with $j < i$. If $\mathcal{B}$ is a special block of λ containing say $b \in [r]$, then we will say that i is a *special block record low* if (i) i occurs to the left of b in $\mathcal{B}$, with no element $j < i$ to the left of i, or (ii) i occurs to the right of b in $\mathcal{B}$, with no $j < i$ occurring between b and i. Let $rec'(\lambda)$ denote the total number of special block record lows in all of its special blocks. Let $nmin(\lambda)$ denote the number of elements of $[r+1, r+n]$ either (a) belonging to a non-special block and not a left-to-right minimum, or (b) belonging to a special block and not a special block record low. For example, if $n = 12$, $k = 2$, $r = 3$, $m = 5$ and

$$\lambda = \{10,7,1,12\},\{2\},\{5,15,3,6,8\},\{4,14,11\},\{13,9\} \in \mathcal{L}_5^{(3)}(12,2),$$

then $rec'(\lambda) = 5$ (the enumerated elements being 10, 7, 12, 5 and 6) and $nmin(\lambda) = 4$ (the elements being 15, 8, 14 and 11). Note that minimal elements in all blocks and left-to-right minima in non-special blocks are among those excluded from the counts of both statistics, whence $0 \leq nmin(\lambda) + rec'(\lambda) \leq n-k$ with all values in this range being realized. Define the joint distribution polynomial for the $nmin$ and rec' statistics on $\mathcal{L}_m^{(r)}(n,k)$ by

$$\ell_m^{(r)}(n,k;a,b) = \sum_{\lambda\in\mathcal{L}_m^{(r)}(n,k)} a^{nmin(\lambda)} b^{rec'(\lambda)}.$$

See [13] for a related generalization of the Lah numbers.

Let $[a,b]_j = \prod_{j=1}^m (aj+b)$ if $j \geq 1$, with $[a,b]_0 = 1$. Considering whether or not the element $n+r$ forms its own block within a member of $\mathcal{L}_m^{(r)}(n,k)$, and if not, considering further cases based on whether $n+r$ follows directly a member of $[r+1, r+n-1]$ or starts a non-special block or is a special block record low yields the recurrence

$$\ell_m^{(r)}(n,k;a,b) = \ell_m^{(r)}(n-1,k-1;a,b) + (a(n-1)+k+2br)\ell_m^{(r)}(n-1,k;a,b)$$

$$
\begin{aligned}
&- [a,1]_m \binom{n-1}{m} \ell_m^{(r)}(n-m-1,k-1;a,b) \\
&- 2br[a,2b]_{m-1} \binom{n-1}{m-1} \ell_m^{(r-1)}(n-m,k;a,b), \qquad (5.1)
\end{aligned}
$$

which reduces to (3.1) when $a = b = 1$. Note that it is possible to consider a further polynomial generalization wherein the $k\ell_m^{(r)}(n-1,k;a,b)$ term in (5.1) is multiplied by an indeterminate c. However, the statistic marked by c in this case can be obtained as $n - k - nmin(\lambda) - rec'(\lambda)$ for all λ. Thus, we may assume without loss of generality that one of a, b or c equals 1, and it is most convenient here to take $c = 1$.

Several of the properties shown in prior sections can be extended to the polynomial case. For example, generalizing the arguments used to show (3.2) and (3.8) respectively yields

$$
\begin{aligned}
\ell_m^{(r)}(n,k;a,b) = n! \sum_{i=0}^{k} \sum_{j=0}^{r} &\frac{(2b)^j [a,1]_{m-1}^i [a,2b]_{m-2}^j}{i!(m!)^i[(m-1)!]^j(n-mi-(m-1)j)!} \binom{r}{j} \\
&\times \ell_{m-1}^{(r-j)}(n-mi-(m-1)j,k-i;a,b) \qquad (5.2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{n} (j+p+2br)[a,j+p+2br]_{k-1} \ell_m^{(r)}(n,j;a,b) \\
&= \sum_{i=0}^{n} \sum_{s=0}^{p} (i+s)! \binom{p}{s} \ell^{(r)}(k,i+s;a,b) L_m^{(r)}(n,i;a,b), \qquad (5.3)
\end{aligned}
$$

where $L_m^{(r)}(n,k;a,b) = \sum_{j=k}^{n} \binom{j}{k} \ell_m^{(r)}(n,j;a,b)$. One can also generalize (3.6) if the second sum in (3.6) is expressed instead using multiple indices which yields

$$
\begin{aligned}
\ell_m^{(r)}(n,k;a,b) = &\sum_{j=1}^{r} (-1)^{j-1} \binom{r}{j} \ell_m^{(r-j)}(n,k;a,b) \\
&+ \sum_{\substack{i_1+\cdots+i_r \le n-k \\ 1 \le i_j \le m-1}} \binom{n}{i_1,\ldots,i_r,n-\sum_{j=1}^{r} i_j} \left((2b)^r \prod_{j=1}^{r} [a,2b]_{i_j-1} \right) \\
&\times \ell_m^{(0)}(n - \sum_{j=1}^{r} i_j, k; a,b). \qquad (5.4)
\end{aligned}
$$

Note that it is possible to write a recurrence for the multi-sum occurring in (5.4) that is analogous to (5.1) above (here, one would need an extra term $2br\ell_m^{(r-1)}(n-1,k;a,b)$ and assume $m \ge 2$).

We conclude by suggesting some further problems to consider. First, it would be interesting to find polynomial generalizations of formulas (3.4) and (4.1). The unimodality of $\ell_m^{(r)}(n,k)$ could be considered as well as for what k the maximum value is achieved when n is fixed. While the r-Lah numbers are log-concave (see, e.g., [16]), it seems to be more difficult to establish this fact for $\ell_m^{(r)}(n,k)$ or, more generally, for $\ell_m^{(r)}(n,k;a,b)$ when a and b are positive real numbers, due to the more complicated formulas that are involved. Finally, it would be interesting to find an orthogonality relation for $\ell_m^{(r)}(n,k)$ that has as its limiting case when $m \to \infty$ the known orthogonality formula for the r-Lah numbers (see [16, Corollary 3.1]).

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